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Elements of a category theory for pre-semiotic relations

1. The elements of a category theory on the basis of the triadic-trichotomic sign relation $SR_{3,3}$ = (3.a 2.b 1.c) have been summarized in Toth (1997, pp. 21 ss.) and continued in Toth (2007). In the present study, I shall show that the category theoretical framework cannot be extended in a trivial way in the transition from the semiotic matrix of $SR_{3,3}$ to the semiotic matrix of $SR_{4,3}$. The non-trivial extension of $SR_{3,3} \rightarrow SR_{4,3}$ has also been shown for the structural realities presented in the reality thematics of the semiotic systems SS10 → SS15 (Toth 2008c).

2. In Toth (2008b), I have introduced the tetradic-trichotomic pre-semiotic sign relation

$SR_{4,3}$ = (3.d 2.c 1.b 0.a)

with the corresponding tetratomic inclusion order

$(d \leq c \leq b \leq a)$, where $a, b, c, d \in \{.1, .2, .3\}$,

whose corresponding semiotic structure is thus 4-adic, but 3-tomic, since in Z^r_k, the categorial number $k \neq 0$ (Bense 1975, p. 65), and therefore, the pre-semiotic matrix is "defective" from the viewpoint of a quadratic matrix of Cartesian products over (.0., .1., .2., .3.). Moreover, we have to introduce semiotic morphisms for the sub-signs (0.1), (0.2), and (0.3), since these are of course not defined in $SR_{3,3}$. We will thus define:

$(0.1) := \gamma$
$(0.2) := \delta$

Therefore, (0.3) can be written as composed morphism $\delta\gamma$ (like (1.3) = $(\beta\alpha)$). For the respective dual morphisms, whose appearance is, naturally, restricted to pre-semiotic reality thematics, we thus get $(0.1)° = (1.0) = \gamma°$; $(0.2)° = (2.0) = \delta°$; $(0.3)° = (3.0) = \gamma°\delta°$ (cf. (3.1) = $(\alpha°\beta°)$). Therefore, we can write the pre-semiotic matrix for $SR_{4,3}$ in the numerical and in the category theoretical way:

	.1	.2	.3
0.	0.1	0.2	0.3
1.	1.1	1.2	1.3
2.	2.1	2.2	2.3
3.	3.1	3.2	3.3

≡

	.1	.2	.3
0.	γ	δ	$\delta\gamma$
1.	id1	α	$\beta\alpha$
2.	$\alpha°$	id2	β
3.	$\alpha°\beta°$	$\beta°$	id3

The above matrices thus show the tetradic-trichotomic dyads of $SR_{4,3}$, or in other words the sub-signs which can appear in the system of the pre-semiotic sign classes of SS15. However, since the dual morphisms of the trichotomies of zeroness cannot appear in the system of the sign classes, we exchange the rows and the columns of the above matrix in order to get the following two matrices which show the tetratomic-triadic dyads of $SR_{4,3}$, or the sub-signs which can appear in the system of the pre-semiotic reality thematics of SS15:

	0.	1.	2.	3.
.1	1.0	1.1	1.2	1.3
.2	2.0	2.1	2.2	2.3
.3	3.0	3.1	3.2	3.3

≡

	0.	1.	2.	3.
.1	γ°	id1	α	βα
.2	δ°	α°	id2	β
.3	γ°δ°	α°β°	β°	id3

Therefore, we can note the complete system of the 15 pre-semiotic sign classes (SS15) in numerical form and in category theoretic manner:

1 (3.1 2.1 1.1 0.1) × (1.0 1.1 1.2 1.3)
[[β°, id1], [α°, id1], [γ°, id1]] × [[id1, γ], [id1, α], [id1, β]]

2 (3.1 2.1 1.1 0.2) × (2.0 1.1 1.2 1.3)
[[β°, id1], [α°, id1], [γ°, α]] × [[α°, γ], [id1, α], [id1, β]]

3 (3.1 2.1 1.1 0.3) × (3.0 1.1 1.2 1.3)
[[β°, id1], [α°, id1], [γ°, βα]] × [α°β°, γ], [id1, α], [id1, β]]

4 (3.1 2.1 1.2 0.2) × (2.0 2.1 1.2 1.3)
[[β°, id1], [α°, α], [γ°, id2]] × [[id2, γ], [α°, α], [id1, β]]

5 (3.1 2.1 1.2 0.3) × (3.0 2.1 1.2 1.3)
[[β°, id1], [α°, α], [γ°, β]] × [[β°, γ], [α°, α], [id1, β]]

6 (3.1 2.1 1.3 0.3) × (3.0 3.1 1.2 1.3)
[[β°, id1], [α°, βα], [γ°, id3]] × [[id3, γ], [α°β°, α], [id1, β]]

7 (3.1 2.2 1.2 0.2) × (2.0 2.1 2.2 1.3)
[[β°, α], [α°, id2], [γ°, id2]] × [[id2, γ], [id2, α], [α°, β]]

8 (3.1 2.2 1.2 0.3) × (3.0 2.1 2.2 1.3)
[[β°, α], [α°, id2], [γ°, β]] × [[β°, γ], [id2, α], [α°, β]]

9 (3.1 2.2 1.3 0.3) × (3.0 3.1 2.2 1.3)

$[[\beta^\circ, \alpha], [\alpha^\circ, \beta], [\gamma^\circ, id3]] \times [[id3, \gamma], [\beta^\circ, \alpha], [\alpha^\circ, \beta]]$

10 (3.1 2.3 1.3 0.3) × (3.0 3.1 3.2 1.3)
$[[\beta^\circ, \beta\alpha], [\alpha^\circ, id3], [\gamma^\circ, id3]] \times [[id3, \gamma], [id3, \alpha], [\alpha^\circ\beta^\circ, \beta]]$

11 (3.2 2.2 1.2 0.2) × (2.0 2.1 2.2 2.3)
$[[\beta^\circ, id2], [\alpha^\circ, id2], [\gamma^\circ, id2]] \times [[id2, \gamma], [id2, \alpha], [id2, \beta]]$

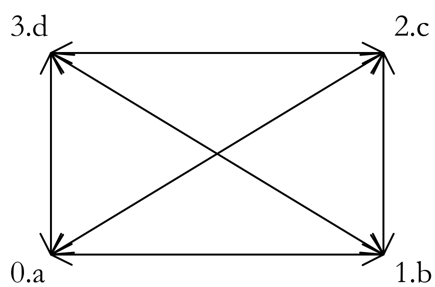
12 (3.2 2.2 1.2 0.3) × (3.0 2.1 2.2 2.3)
$[[\beta^\circ, id2], [\alpha^\circ, id2], [\gamma^\circ, \beta]] \times [[\beta^\circ, \gamma], [id2, \alpha], [id2, \beta]]$

13 (3.2 2.2 1.3 0.3) × (3.0 3.1 2.2 2.3)
$[[\beta^\circ, id2], [\alpha^\circ, \beta], [\gamma^\circ, id3]] \times [[id3, \gamma], [\beta^\circ, \alpha], [id2, \beta]]$

14 (3.2 2.3 1.3 0.3) × (3.0 3.1 3.2 2.3)
$[[\beta^\circ, \beta], [\alpha^\circ, id3], [\gamma^\circ, id3]] \times [[id3, \gamma], [id3, \alpha], [\beta^\circ, \beta]]$

15 (3.3 2.3 1.3 0.3) × (3.0 3.1 3.2 3.3)
$[[\beta^\circ, id3], [\alpha^\circ, id3], [\gamma^\circ, id3]] \times [[id3, \gamma], [id3, \alpha], [id3, \beta]]$

3. Since the pre-semiotic sign relation $SR_{4,3}$ = (3.a 2.b 1.c 0.d) has four correlates, we need a square in order to display the pre-semiotic sign model:



This rectangular sign model is a graph with 4 vertices and 6 edges. The 6 edges correspond to sign functions, which had been defined for the sign model of $SR_{3,3}$ = (3.c 2.b 1.a) or its corresponding graph with 3 vertices and 3 edges, respectively, by Walther (1979, pp. 113 s.):

(1.c ⇒ 2.b) "designation function"
(2.b ⇒ 3.a) "denomination function"
(3.a ⇒ 1.c) "utilization function"

However, if we generalize these semiotic functions, we get the following dual system of pre-semiotic sign functions:

1.	(0.a ⇒ 1.b)	1.'	(1.b ⇒ 0.a)	5.	(2.c ⇒ 0.a)
2.	(1.b ⇒ 2.c)	2.'	(2.c ⇒ 1.b)	5'.	(0.a ⇒ 2.c)

3.	$(2.c \Rightarrow 3.d)$	3.’	$(3.d \Rightarrow 2.c)$	6.	$(3.d \Rightarrow 1.b)$
4.	$(3.d \Rightarrow 0.a)$	4.’	$(0.a \Rightarrow 3.d)$	6.’	$(1.b \Rightarrow 3.d)$

Since here, dyadic sub-signs are mapped to pairs of dyadic subsigns, we can apply dynamic morphisms (cf. Toth 2008a, pp. 159 ss.) and define them as follows:

((0.a), (1.b)) := [[0.1], [a.b]] ((1.b), (0.a)) := [[1.0], [b.a]] ((2.c), (0.a)) := [[2.0], [c.a]]
((1.b), (2.c)) := [[1.2], [b.c]] ((2.c), (1.b)) := [[2.1], [c.b]] ((0.a), (2.c)) := [[0.2], [a.c]]
((2.c), (3.d)) := [[2.3], [c.d]] ((3.a), (2.c)) := [[3.2], [a.c]] ((3.d), (1.b)) := [[3.1], [d.b]]
((3.d), (0.a)) := [[3.0], [d.a]] ((0.a), (3.d)) := [[0.3], [a.d]] ((1.b), (3.d)) := [[1.3], [b.d]]

Now, let us take the three pre-semiotic sign classes

1. (3.1 2.1 1.2 0.2), i.e. a = 2, b = 2, c = 1, d = 1
2. (3.1 2.1 1.3 0.3), i.e. a = 3, b = 3, c = 1, d = 1
3. (3.2 2.2 1.2 1.3), i.e. a = 3, b = 2, c = 2, d = 2

1. Then, we get for the 1st pre-semiotic sign class:

((0.3), (1.2)) := [[0.1], [3.2]] ≡ [δγ, α] ((2.2), (0.3)) := [[2.0], [2.3]] ≡ [α°, δ]
((1.2), (2.2)) := [[1.2], [2.2]] ≡ [α, α°] ((0.3), (2.2)) := [[0.2], [3.2]] ≡ [δ, α°]
((2.2), (3.2)) := [[2.3], [2.2]] ≡ [α°, α°β°] ((3.2), (1.2)) := [[3.1], [2.2]] ≡ [α°β°, α]
((3.2), (0.3)) := [[3.0], [2.3]] ≡ [α°β°, δ] ((1.2), (3.2)) := [[1.3], [2.2]] ≡ [α, α°β°]

((1.2), (0.3)) := [[1.0], [2.3]] ≡ [α, δ]
((2.2), (1.2)) := [[2.1], [2.2]] ≡ [α°, α]
((3.2), (2.2)) := [[3.2], [2.2]] ≡ [β°, α°]
((0.3), (3.2)) := [[0.3], [3.2]] ≡ [δ, α°β°]

a) $(0.2 \Rightarrow 1.2) \circ (1.2 \Rightarrow 2.1) = (0.2 \Rightarrow 2.1) \equiv [\gamma, id2] \circ [\alpha, \alpha°] = [\delta, \alpha°]$
b) $(2.1 \Rightarrow 3.1) \circ (3.1 \Rightarrow 0.2) = (2.1 \Rightarrow 0.2) \equiv [\beta, id1] \circ [\gamma°\delta°, \alpha] = [\delta°, \alpha]$
c) $(3.1 \Rightarrow 2.1) \circ (2.1 \Rightarrow 1.2) = (3.1 \Rightarrow 1.2) \equiv [\beta°, id1] \circ [\alpha°, \alpha] = [\alpha°\beta°, \alpha]$
d) $(3.1 \Rightarrow 0.2) \circ (0.2 \Rightarrow 1.2) = (3.1 \Rightarrow 1.2) \equiv [\gamma°\delta°, \alpha] \circ [\gamma, id2] = [\alpha°\beta°, \alpha]$

From a), it follows that $\gamma \circ \alpha = \delta$; $id2 \circ \alpha° = \alpha°$.
From b), it follows that $\beta \circ \gamma°\delta° = \delta°$; $id1 \circ \alpha = \alpha$.
From c), it follows that $\beta° \circ \alpha° = \alpha°\beta°$; $id1 \circ \alpha = \alpha$.
From d), it follows that $\gamma°\delta° \circ \gamma = \alpha°\beta°$; $\alpha \circ id2 = \alpha$.

2. We get for the 2nd pre-semiotic sign class:

((0.3), (1.3)) := [[0.1], [3.3]] ≡ [γ, id3] ((2.1), (0.3)) := [[2.0], [1.3]] ≡ [δ°, βα]
((1.3), (2.1)) := [[1.2], [3.1]] ≡ [βα, α°] ((0.3), (2.1)) := [[0.2], [3.1]] ≡ [δ, α°β°]
((2.1), (3.1)) := [[2.3], [1.1]] ≡ [α°, α°β°] ((3.1), (1.3)) := [[3.1], [1.3]] ≡ [α°β°, βα]
((3.1), (0.3)) := [[3.0], [1.3]] ≡ [α°β°, δγ] ((1.3), (3.1)) := [[1.3], [3.1]] ≡ [βα, α°β°]

$((1.3), (0.3)) := [[1.0], [3.3]] \equiv [\beta\alpha, \delta\gamma]$
$((2.1), (1.3)) := [[2.1], [1.3]] \equiv [\alpha°, \beta\alpha]$
$((3.2), (2.1)) := [[3.2], [2.1]] \equiv [\beta°, \alpha°]$
$((0.3), (3.1)) := [[0.3], [3.1]] \equiv [\delta\gamma, \alpha°\beta°]$

a) $(0.3 \Rightarrow 1.3) \circ (1.3 \Rightarrow 2.1) = (0.3 \Rightarrow 2.1) \equiv [\gamma, id3] \circ [\alpha, \alpha°\beta°] = [\delta, \alpha°\beta°]$
b) $(2.1 \Rightarrow 3.1) \circ (3.1 \Rightarrow 0.3) = (2.1 \Rightarrow 0.3) \equiv [\beta, id1] \circ [\gamma°\delta°, \beta\alpha] = [\delta°, \beta\alpha]$
c) $(3.1 \Rightarrow 2.1) \circ (2.1 \Rightarrow 1.3) = (3.1 \Rightarrow 1.3) \equiv [\beta°, id1] \circ [\alpha°, \beta\alpha] = [\alpha°\beta°, \beta\alpha]$
d) $(3.1 \Rightarrow 0.3) \circ (0.3 \Rightarrow 1.3) = (3.1 \Rightarrow 1.3) \equiv [\gamma°\delta°, \beta\alpha] \circ [\gamma, id3] = [\alpha°\beta°, \beta\alpha]$

From a), it follows that $\gamma \circ \alpha = \delta$; $id3 \circ \alpha°\beta° = \alpha°\beta°$.
From b), it follows that $\beta \circ \gamma°\delta° = \delta°$; $id1 \circ \beta\alpha = \beta\alpha$.
From c), it follows that $\beta° \circ \alpha° = \alpha°\beta°$; $id1 \circ \beta\alpha = \beta\alpha$.
From d), it follows that $\gamma°\delta° \circ \gamma = \alpha°\beta°$; $\beta\alpha \circ id3 = \beta\alpha$.

3. We get for the 3rd pre-semiotic sign class:

$((0.3), (1.2)) := [[0.1], [3.2]] \equiv [\gamma, \beta°]$
$((1.2), (2.2)) := [[1.2], [2.2]] \equiv [\alpha, id2]$
$((2.2), (3.2)) := [[2.3], [2.2]] \equiv [id2, \beta°]$
$((3.2), (0.3)) := [[3.0], [2.3]] \equiv [\beta°, \delta\gamma]$

$((2.2), (0.3)) := [[2.0], [2.3]] \equiv [\delta°, \beta]$
$((0.2), (2.2)) := [[0.2], [2.2]] \equiv [\delta, id2]$
$((3.2), (1.2)) := [[3.1], [2.2]] \equiv [\beta°, \alpha]$
$((1.2), (3.2)) := [[1.3], [2.2]] \equiv [\alpha, \beta°]$

$((1.2), (0.3)) := [[1.0], [2.3]] \equiv [\gamma°, \beta]$
$((2.2), (1.2)) := [[2.1], [2.2]] \equiv [\alpha°, id2]$
$((3.2), (2.2)) := [[3.2], [2.2]] \equiv [\beta°, id2]$
$((0.3), (3.2)) := [[0.3], [3.2]] \equiv [\delta\gamma, \beta°]$

a) $(0.3 \Rightarrow 1.2) \circ (1.2 \Rightarrow 2.2) = (0.3 \Rightarrow 2.2) \equiv [\gamma, \beta°] \circ [\alpha, id2] = [\delta, \beta°]$
b) $(2.2 \Rightarrow 3.2) \circ (3.2 \Rightarrow 0.3) = (2.2 \Rightarrow 0.3) \equiv [\beta, id1] \circ [\gamma°\delta°, \beta] = [\delta°, \beta]$
c) $(3.2 \Rightarrow 2.2) \circ (2.2 \Rightarrow 1.2) = (3.2 \Rightarrow 1.2) \equiv [\beta°, id2] \circ [\alpha°, id2] = [\alpha°\beta°, id2]$
d) $(3.2 \Rightarrow 0.3) \circ (0.3 \Rightarrow 1.2) = (3.2 \Rightarrow 1.2) \equiv [\gamma°\delta°, \beta] \circ [\gamma, \beta°] = [\alpha°\beta°, id2]$

From a), it follows that $\gamma \circ \alpha = \delta$; $\beta° \circ id2 = \beta°$.
From b), it follows that $\beta \circ \gamma°\delta° = \delta°$; $id1 \circ \beta = \beta$.
From c), it follows that $\beta° \circ \alpha° = \alpha°\beta°$; $id2 \circ id2 = id2$.
From d), it follows that $\gamma°\delta° \circ \gamma = \alpha°\beta°$; $\beta \circ \beta° = id2$.

4. From the above three representative examples, we recognize the following rules of computing dynamic semiotic morphisms (which must NOT be confused with the rules of computing “regular” category theoretic morphisms!):

1. $X \circ idx = idx \circ X = X$, where $X \in \{\alpha, \alpha°, \beta, \beta°, \beta\alpha, \alpha°\beta°, id1, id2, id3\}$ and $x \in \{1, 2, 3\}$
2. $X \circ X° = idx$, where $x \in \{1, 2, 3\}$ (!)
3. (a) $\gamma \circ \alpha = \delta \equiv ((0.1\ 1.2) = (0.2))$
   (b) $\beta \circ \gamma°\delta° = \delta° \equiv ((2.3\ 3.0) = (2.0)$
   (c) $\beta° \circ \alpha° = \alpha°\beta° \equiv ((3.2\ 2.1) = (3.1))$
   (d) $\gamma°\delta° \circ \gamma = \alpha°\beta°$ ($\Rightarrow \beta°\alpha° = \gamma°\delta°\gamma \equiv ((3.0\ 0.1) = (3.1))$

Thus, these results, that look very strange from the standpoint of "regular" category theory, show how the pre-semiotic sign classes deal with the above mentioned "defectivity" of the non-square 4 × 3 matrix of $SR_{4,3}$, in which there are neither dual, nor composed, nor composed dual equivalents of the morphisms γ and δ: They are being substituted by (dual) compositions of one of the morphisms γ or δ plus the morphisms α or β or their (dual) compositions. Moreover, in pre-semiotic SS15, the dual composition $\alpha°\beta°$ can be reached in two ways: First, by the composition $\beta° \circ \alpha°$, which does not use γ or δ, but which is still not defined in SS10, and second, by the composition $\gamma°\delta° \circ \gamma$.

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