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## Semiotic calculi of extensional sign-connections

1. In this paper, I shall continue some ideas adopted for semiotics in Toth (2008b, c). I intend to show that Clarke's "Calculus of Individuals" (CI), which is based on the Whiteheadean primitive binary predicate "x is extensionally connected with y", is valid for semiotics, too. Insofar, this paper is also a sequel of my mathematical-logic semiotics presented in Toth (2007, pp. 143 ss.). As for the parts of this study, I will also follow Clarke (1981), who subdivided his landmark-study into a mereological, a quasi-Boolean (without null element), and a quasi-topological (without boundary elements) part.

2. Mereological calculus

We define the following first-order operators:

Cx,y = x is connected to y
DCx,y = x is disconnected from y
Px,y = x is a part of y
PPx,y = x is a proper part of y
Ox,y = x overlaps y (i.e., x and y share a common interior point)
DRx,y = s is discrete from y
ECx,y = x is externally connected to y
TPx,y = x is a tangential part of y
NTPx,y = x is a non-tangential part of y

as follows and illustrate them with semiotic examples:

D0.1 DCx,y := ¬C,xy
E.g., DC(1.2, 1.3) = ¬C(1.2, 1.3)

D0.2 Px,y := ∀z (Cz,x → Cz,y)
E.g., P(x, y) = (1.3) → (2.2 1.3)

D0.3 PPx,y := Px,y ∧ ¬Py,x
E.g., (1.3) → (2.2 1.3) ∧ ¬((2.2 1.3) → (1.3))

D0.4 Ox,y := ∃z (Pz,x ∧ Pzy)
E.g., (3.1 2.2) ∧ (2.2 1.3) = (1.3)

D0.5 DRx,y := ¬Ox,y
E.g., (3.1 1.1) ∧ (2.2 1.3) = ∅

D0.6 $EC x,y := C x,y \wedge \neg O x,y$

E.g., x = (3.1 2.1 1.1) and y = (3.2 2.2 1.2) are connected, since (3.1) < (3.2), (2.1) < (2.2), (1.1) < (1.2), (2.1 1.1) < (2.2 1.2), and (3.1 2.1 1.1) < (3.2 2.2 1.2), but x does not overlap y, since (3.1 2.1 1.1) ∧ (3.2 2.2 1.2) = ∅.

D0.7 $TP x,y := P x,y \wedge \exists z\, (EC z,x \wedge EC z,y)$

E.g., x = (3.2 2.2 1.2), y = (3.3 2.3 1.3). If we assume that z = (3.1 2.1 1.1), then P(x,y) holds, because (3.2 2.2 1.2) < (3.3 2.3 1.3), and since (3.1 2.1 1.1) < (3.2 2.2 1.2) and also (3.1 2.1 1.1) < (3.3 2.3 1.3), it follows that (3.2 2.2 1.2) is a tangential part of (3.3 2.3 1.3).

D0.8 $NTP x,y := P xy \wedge \neg\exists z\, (EC z,x \wedge EC z,y)$

E.g., x = (3.1 2.1 1.1), y = (3.2 2.2 1.2), and z = (3.3 2.3 1.3). Although (3.1 2.1 1.1) < (3.2 2.2 1.2), (3.3 2.3 1.3) is neither externally connected to x nor to y, so that x is a non-tangential part of y.

Clarke's axiomatization requires only the following to axioms, a mereological axiom and an axiom analogous to the axiom of extension in set theory (Clarke 1981, p. 206):

A0.1 $\forall x[C x,x \wedge \forall y\, (C x,y \rightarrow C y,x)]$

E.g., for each x, y ∈ {1, 2, 3} (the set of the prime-signs, cf. Bense 1980), there is the set {(1.1), (2.2), (3.3)}, and for each sub-signs of the structure (a.b), there is also the corresponding sub-signs of the structure (b.a) in the semiotic matrix, i.e. for (1.2), there is (2.1), for (1.3), there is (3.1), and for (2.3), there is (3.2) in the semiotic matrix. In other words, AO.1 alone is sufficient to construct all the sub-signs of the semiotic matrix.

A0.2 $\forall x\, \forall y\, [\forall z\, (C z,x \equiv C z,y) \rightarrow x = y]$

E.g., let us assume that there are two semiotic sets S = {(1.1), (1.2), (1.3), (2.1), (2.2), (2.3), (3.1), (3.2), (3.3)} and S' = {(1.1'), (1.2'), (1.3'), (2.1'), (2.2'), (2.3'), (3.1'), (3.2'), (3.3')}. Then, AO.2 says S ≡ S', iff they have precisely the same members. In other words, any set is determined uniquely by its members.

The following 47 theorems that are based on the two axioms and the eight definitions given above, are displayed here in the order of Clarke (1981, pp. 206 s.):

T0.1 $\forall x\, C x,x$

E.g., take any x ∈ = {(1.1), (1.2), (1.3), (2.1), (2.2), (2.3), (3.1), (3.2), (3.3)}.

T0.2 $\forall x \forall y\, (C x,y \equiv C y,x)$

E.g., if (2.1) is connected to (3.1), then (3.1) is also connected to (2.1).

T0.3 $\forall x \forall y\ [\forall z\ (Cz,x = Cz,y) \equiv x = y]$

E.g., if x = (2.1), y = (2.1'), and z = (3.1), then to say that (3.1, 2.1) = (3.1, 2.1') is identical to say that (2.1) = (2.1').

T0.4 $\forall x \forall y\ (\neg DCx,y \equiv Cx,y)$

E.g., if two sub-signs are not disconnected, then they must be connected.

T0.5 $\forall x\ Px,x$

E.g., (1.3) ≤ (1.3).

T0.6 $\forall x \forall y \forall z\ [(Px,y \wedge Py,z) \rightarrow Px,z]$

E.g., if (3.1) is a part of (3.1 2.2), and (3.1 2.2) is a part of (3.1 2.2 1.3), then (3.1) is also a part of (3.1 2.2 1.3).

T0.7 $\forall x \forall y\ [(Px,y \wedge Py,x) \equiv x = y]$

E.g., if (3.1) is a part of (3.1 2.2), and (3.1 2.2) is a part of (3.1), then (3.1) = (3.1 2.2), which is wrong.

T0.8 $\forall x \forall y\ [Px,y \equiv \forall z\ (Pz,x \rightarrow Pz,y)]$

E.g., if (3.1) is a part of (3.1 2.2), then ((a.b), 3.1)) → ((a.b), (3.1 2.2)) for (a.b) ∈ = {(1.1), (1.2), (1.3), (2.1), (2.2), (2.3), (3.1), (3.2), (3.3)}.

T0.9 $\forall x \forall y \forall z\ [(Px,y \wedge Cz,x) \rightarrow Cz,y]$

E.g., if x = (1.2) and y = (1.3), then P((1.2), (1.3)). If z = (1.1), then C((1.1), (1.2), and it follows that C((1.1), (1.3)).

T0.10 $\forall x \forall y\ [Cx,y \equiv \exists z\ (Pz,y \wedge Cx,z)]$

E.g., if x = (1.2), y = (1.3), and z = (1.1), then C((1.2), (1.3)) ≡ (P((1.1), (1.3)) ∧ C((1.2), (1.1)).

T0.11 $\forall x \forall y\ (Px,y \rightarrow Cx,y)$

E.g., P((1.2), (1.2 2.2)) → C((1.2), (1.2 2.2))

T0.12 $\forall x \forall y \forall z\ [(Px,y \wedge DCz,y) \rightarrow DCz,x]$

E.g., if P((2.1), (2.1 3.1)) ∧ DC((1.1), (2.1 3.1)) → DC((1.1), (2.1)).

T0.13 $\forall x \neg PPx,x$

E.g., PP((1.3), (1.3)).

T0.14 $\forall x \forall y\ (PPx,y \rightarrow Px,y)$

E.g., PP((1.3), (1.3 2.2)) $\rightarrow$ P((1.3), (1.3 2.2)).

T0.15 $\forall x \forall y\ (PPx,y \rightarrow \neg PPy,x)$

E.g., PP((1.3), (1.3 2.2)) $\rightarrow \neg$PP ((1.3 2.2), (1.3)).

T0.16 $\forall x \forall y \forall z\ [(PPx,y \wedge PPy,z) \rightarrow PPx,z]$

E.g., PP((1.3), (1.3 2.2)) $\wedge$ ((1.3 2.2), (1.3 2.2 3.1)) $\rightarrow$ PP((1.3), (1.3 2.2 3.1))

T0.17 $\forall x\ Ox,x$

E.g., (1.2) $\wedge$ (1.2) = (1.2).

T0.18 $\forall x \forall y\ (Ox,y = Oy,x))$

E.g., O((1.2), (2.2 1.2)) = O((2.2 1.2), (1.2)).

T0.19 $\forall x \forall y\ (Ox,y \rightarrow Cx,y)$

E.g., O((1.2), (2.2 1.2)) $\rightarrow$ C((1.2), (2.2 1.2)).

T0.20 $\forall x \forall y\ [(Px,y \wedge Oz,x) \rightarrow Oz,y]$

E.g., P((1.2), (2.2 1.2)) $\wedge$ O((2.2 1.3), (1.2)) $\rightarrow$ O((2.2 1.3), (2.2 1.2)).

T0.21 $\forall x \forall y\ (Px,y \rightarrow Ox,y)$

E.g., P((1.2), (2.2 1.2)) $\rightarrow$ O((1.2), (2.2 1.2))

T0.22 $\forall x \forall y\ (\neg DRx,y \equiv Ox,y)$

E.g., $\neg$DR((1.2), (2.2 1.2)) $\equiv$ O((1.2), (2.2 1.2))

T0.23 $\forall x \forall y \forall z\ [(Px,y \wedge DRz,y) \rightarrow DRz,x]$

E.g., P((1.2), (2.2 1.2)) $\wedge$ DR((3.3), (2.2 1.2)) $\rightarrow$ DR((3.3), (1.2)).

T0.24 $\forall x \neg ECx,x$

E.g., any sub-sign is externally connected with itself; i.e., the intersection of a sub-sign with itself is non-empty.

T0.25 $\forall x \forall y\ (ECx,y \equiv ECy,x)$

E.g., if a sign x which is externally connected to a sign y, then y is also externally connected to x.

T0.26 $\forall x \forall y\ (ECx,y \rightarrow Cx,y)$

E.g., every sign that is externally connected is also connected.

T0.27 $\forall x \forall y\ (ECx,y \rightarrow \neg Ox,y)$

E.g., every sign x that is externally connected to a sign y, does not overlap with y, i.e. x and y do not share internal parts.

T0.28 $\forall x \forall y\ [Cx,y \equiv (ECx,y \vee Ox,y)]$

E.g., two signs x and y are connected iff they are either externally connected or overlap.

T0.29 $\forall x \forall y\ [Ox,y \equiv (Cx,y \wedge \neg ECx,y)]$

E.g., a sign x overlaps a sign y iff x and y are connected, but not externally connected.

T0.30 $\forall y \forall y\ [\neg ECx,y \equiv (Ox,y \equiv Cx,y)]$

E.g., if two signs x and y are not externally connected, then x overlaps y, and x is connected to y.

T0.31 $\forall y \forall y\ [\neg \exists z\ ECz,x \rightarrow [Px,y \equiv \forall z\ (Oz,x \rightarrow Oz,y)]]$

E.g., if there is not sign z, which is externally connected to x, than it follows that $[Px,y \equiv \forall z\ (Oz,x \rightarrow Oz,y)]]$, cf. D0.2.

T0.32 $\forall x \forall y\ (TPx,y \rightarrow P(x,y)$

E.g., if a sign x is a tangential part of a sign y, then x is a part of y.

T0.33 $\forall x \forall y\ [TPx,y \rightarrow \exists z\ (ECz,x \wedge ECz,y)]$

E.g., If a sign x is a tangential part of a sign y, then there a sign z, so that z is both connected to x and to y.

T0.34 $\forall x \forall y \forall z\ [(TPz,x \wedge Pz,y \wedge Py,x) \rightarrow TPz,y]$

E.g., if a sign z is a tangential part of a sign x, and z is a part of a sign y, and y is a part of x, then z is a tangential part of y.

T0.35 $\forall x \forall y\ (NTPx,y \rightarrow Px,y)$

E.g., If a sign x is a non-tangential part of a sign y, then x is a part of y.

T0.36 $\forall x \forall y\ [NTPx, y \rightarrow \neg \exists z\ (ECz,x \wedge ECz,y)]$

E.g., if a sign x is a non-tangential part of a sign y, then there is not sign z, to which both x and y are externally connected.

T0.37 $\forall x \forall y\ (TPx,y \rightarrow \neg NTPx,y)$

E.g., if a sign x is a tangential part of a sign y, then x cannot be at the same time a non-tangential part of y.

T0.38 $\forall x \forall y\ [TPx,y \equiv (Px,y \wedge \neg NTPx,y)]$

E.g., if a sign x that is a part of a sign y, and x is not a non-tangential part of y, then x is a tangential part of y.

T0.39 $\forall x \forall y\ [NTPx,y \equiv (Px,y \wedge \neg TPx,y)]$

E.g., If a sign x is a part of a sign y, and x is not a tangential part of y, then it is a non-tangential part.

T0.40 $\forall x \forall y\ [Px,y \equiv (TPx,y \vee NTPx,y)]$

E.g., A sign x that is part of a sign y, is either a tangential or a non-tangential part of y.

T0.41 $\forall x\ (NTPx,x \equiv \neg \exists y\ ECy,x)$

E.g., a sign x is a non-tangential part of itself, iff there is is no sign y that is externally connected to x.

T0.42 $\forall x \forall y \forall z\ [(NTPx,y \wedge Cz,x) \rightarrow Cz,y]$

E.g., If a sign x is a non-tangential part of a sign y, and a sign z is connected to x, then z is also connected to y.

T0.43 $\forall y \forall y \forall z\ [(NTPx,y \wedge Oz,x) \rightarrow Oz,y]$

E.g., if a sign x is a non-tangential part of a sign y, and if a sign z overlaps x, then z overlaps y, too.

T0.44 $\forall x \forall y \forall z\ [(NTPx,y \wedge Cz,x) \rightarrow Oz,y]$

E.g., if a sign x is a non-tangential part of a sign y, and if a sign z is connected to x, then z overlaps y, too.

T0.45 $\forall x \forall y \forall z\ [(Px,y \wedge NTPy,z) \rightarrow NTPx,z]$

E.g., if a sign x is a part of the sign y, and y is a non-tangential part of the sign z, then x (too) is a non-tangential part of z.

T0.46 $\forall x \forall y \forall z\ [(NTPx,y \wedge Py,z) \rightarrow NTPx,z]$

E.g., if a sign x is a non-tangential part of a sign y, and y is a part of a sign z, then x is a non-tangential part of z (, too).

T0.47 $\forall x \forall y \forall z\ [(NTPx,y \wedge NTPy,z) \rightarrow NTPx,z]$

E.g., if a sign x is a non-tangential part of a sign y, and y is a non-tangential part of a sign z, then x is a non-tangential part of a sign z.

3. Quasi-Boolean calculus

Following Clarke (1981, pp. 208 ss.), X, Y, and Z are taken as variables ranging over sets of individuals, that is, sub-sets of $\{x: Cx,x\}$. The expression $x = f\text{'}X$ means "x is identical to the fusion of the set X":

D1.1 $x = f\text{'}X := \forall y\ [Cy,x \equiv \exists z\ (z \in X \wedge Cy,z)]$

Using this definition, we shall define x + y for the quasi-Boolean union, -x for the quasi-Boolean complement, a* for the quasi-Boolean universal, and $x \wedge y$ for the quasi-Boolean intersection:

D1.2 $x + y := f\text{'}[z: Pz,x \vee Pz,y]$
D1.3 $-x := f\text{'}[y: \neg Cy,x]$
D1.4 $a^* := f\text{'}[y: Cy,y]$
D1.5 $x \wedge y := f\text{'}[z: Pz,x \cdot Pz,y]$

In addition to these definitions, we need the following axiom:

A1.1 $\forall X\ (\neg X = \Lambda \rightarrow \exists x\ x = f\text{'}X)$

In order to display the 47 theorems built on this axiom and the definitions, we follow again Clarke (1981, pp. 209 ss.):

T1.1 $\forall X \{\neg X = \Lambda \rightarrow \forall x\, [Cx,f'X \equiv \exists y\, (y \in X \wedge Cx,y)]\}$

T1.2. $\forall x\, (\neg X = \Lambda \equiv \exists x\; x = f'X)$

T1.3 $\forall X \forall x\, (x \in X \rightarrow Px,f'X)$

T1.4 $\forall X \forall Y\, [\neg X = \Lambda \wedge X \subseteq Y) \rightarrow Pf'X,f'Y]$

T1.5 $\forall X \forall Y\, [\neg X = \Lambda \wedge X = Y) \rightarrow f'X = f'Y$

T1.6 $\forall x\; x = f'\, \{x)$

T1.7 $\forall x\; x = f'\, \{y: Py,x\}$

T1.8 $\forall x\; f'\, \{x\} = f'\, \{y: Py,x\}$

We explain here T1.1 – T.18 together. X can be defined in several ways, f.ex. as the set of the monadic sub-signs, X = {a.1, b.1, c.1}, the set of dyadic sub-signs, X = {a.2, b.2, c.2}, or the set of triadic sub-signs, X = {a.3, b.3, c.3}, where a, b, c ∈ {1., 2., 3.}. Alternatively, X can be defined as the trichotomy of firstness, X = {1.a, 1.b, 1.c}, as the trichotomy of secondness, X = {2.a, 2.b, 2.c}, or as the trichotomy of thirdness, X = {3.a, 3.b, 3.c}, or, e.g. as the set of diagonal sub-signs, then X = {1.1, 2.2, 3.3} or X = {3.1, 2.1, 1.3}, etc. E.g., if X = {1.a, 1.b, 1.c}, then ¬X = Λ = {2.a, 2.b, 2.c, 3.a, 3.b, 3.c}.

T1.9 $\forall x \forall y\, \exists z\; z = x + y$

E.g., (1.1) + (1.2) = ((1.1), (1.2)) (cf. Toth 2007, p. 144). For the union of sign-classes and reality thematics cf. Berger (1976).

T1.10 $\forall x \forall y \forall z\, \{Cz,x+y \equiv \exists w\, [Pw,x \vee Pw,y) \wedge Cz,w]\}$

E.g., two sub-signs z = (1.1), and the sum (1.1) + (1.2) = (1.3) are connected, means the same as that there is a w such that w is a part of x, or w is a part of y, and z and w are connected.

T1.11 $\forall x \forall y \forall z\, [Cz,x+y \equiv (Cz,x \vee Cz,y)]$

E.g., let be z = (1.1), x = (1.1), y = (1.2), then z and (x + y) are connected means, that either z and x are connected, or z and y are connected.

T1.12 $\forall X \forall Y\, [\neg X = \Lambda \wedge \neg Y = \Lambda) \rightarrow f'X \vee Y = f'X + f'Y]$

Cf. T1.8.

T1.13 $\forall x \forall y\ x + y = f'\{x\} \vee \{y\}$

Cf. T1.8.

T1.14 $\forall x\ x + x = x$

E.g., (1.2) + (1.2) = (1.2), or generally (a.b) + (a.b) = (a.b).

T1.15 $\forall x \forall y\ x + y = y + x$

E.g., (1.2) + (1.3) = (1.3) + (1.2).

T1.16 $\forall x \forall y \forall z\ (x + y) + z = x + (y + z)$

E.g. (1.1 + 1.2) + 1.3 = 1.1 + (1.2 + 1.3).

T1.17 $\forall x \forall y P x, x + y$

E.g. if x = (1.1) and x + y = (1.1 + 1.2), then P((1.1), (1.1 + 1.2)).

T1.18 $\forall x \forall y \forall z\ [(Pz,x \vee Pz,y) \rightarrow Pz,x+y]$

E.g., P((1.1), (1.3)) $\vee$ P((1.1), (1.2)) $\rightarrow$ P((1.3), (1.3 + 1.2)).

T1.19 $\forall x \forall y \forall z\ (Px,y \rightarrow Px,y+z)$

E.g., P((1.2), (1.2 3.2)) $\rightarrow$ P((1.2), ((1.2 3.2 2.3)).

T1.20 $\forall x \forall y \forall z\ (x = y \rightarrow z + x = z + y)$

E.g., if a sign x is even with a sign y, then the union of the sign x any a sign z is even to the union of the sign y and the sign z.

T1.21 $\forall x \forall y\ (Px,y \equiv Px+y, y)$

E.g., P((1.2), (1.2 2.2)) $\equiv$ P((1.2 + (1.2 2.2)), (1.2 2.2)).

T1.22 $\forall x \forall y\ (Px,y) \equiv y = x + y$

E.g., P((1.2, (1.2 2.2)) $\equiv$ (1.2 2.2) = (1.2 + 1.2 2.2).

T1.23 $\exists x\ x = a^*$

Cf. A1.1

T1.24 $\forall x\ [Cx,a^* \equiv \exists y\ (Cy,y \wedge Cx,y)]$

E.g., if both x and y als elements of the set of prime-signs {1, 2, 3}, then $Cy,y \wedge Cx,y$ already scoops out all the elements of the set of sub-signs {(1.1), (1.2), (1.3), (2.1), (2.2), (2.3), (3.1), (3.2), (3.3)}.

T1.25 $\forall x\ Px,a^*$

Cf. T1.24. Since $a^* = \{y: Cy,y\}$ (D1.4), each $x \in$ signs {(1.1), (1.2), (1.3), (2.1), (2.2), (2.3), (3.1), (3.2), (3.3)} is a part of a*.

T1.26 $\forall x\ Cx,a^*$

According to T0.11, we have: $\forall x \forall y\ (Px,y \rightarrow Cx,y)$, and since the set of sub-signs fulfills $\forall x\ Px,a^*$ (T1.25), this implies T1.26.

T1.27 $\forall x\ Ox,a^*$

Since the set of sub-signs fulfills T1.25 and since we have T0.21: $\forall x \forall y\ (Px,y \rightarrow Ox,y)$, it follows immediately that the set of sub-signs fulfills T1.27, too.

T1.28 $\forall x\ x + a^* = a^*$

Since $a^* = \{y: Cy,y\}$, the union of any $x \in$ signs {(1.1), (1.2), (1.3), (2.1), (2.2), (2.3), (3.1), (3.2), (3.3)} with $a^* = a^*$.

T1.29 $\forall x\ (\forall y\ Py,x \equiv x = a^*)$

Since $a^* = \{y: Cy,y\}$ (D1.4), T1.29 follows immediately from T1.25 and T1.26.

T1.30 $\forall x\ (\forall y\ Cy,x \equiv x = a^*)$

According to T0.11, we have: $\forall x \forall y\ (Px,y \rightarrow Cx,y)$, so T1.30 follows directly from T1.29.

T1.31 $\forall x\ \neg ECx,a^*$

E.g., since $a^* = \{y: Cy,y\}$, the semiotic connection must have internal points.

T1.32 $\forall x\ (\exists y\ y = \neg x \equiv \neg x = a^*$

E.g., let be x = (3.a), then one possible negate is (a.3), where $a \in \{1, 2, 3\}$. However, because of T1.8, it follows for semiotics, that every sub-sign (a.b) can substitute its own negate!

T1.33 $\forall x \{\exists z\ z = -x \rightarrow \forall y\ [Cy,-x \equiv \exists z\ (\neg Cz,x \wedge Cy,z)]\}$

E.g., if z = -x, then the connection Cy,-x excludes the existence of ¬C-x,x, i.e. the connection of a sign x with its complement (cf. Toth 2007, p. 143).

T1.34 $\forall x\ [\exists z\ z = -x \rightarrow \forall y\ (Cy,-x \equiv \neg Py,x)]$

E.g., if two signs y and –x (i.e. the complement of x) are connected, then y cannot be a part of x.

T1.35 $\forall x\ (\exists z\ z = -x \rightarrow x = --x$

E.g., the complement of the complement (or the negate of the negative, respectively) of a sign is the sign itself.

T1.36 $\forall x\ [\exists z\ z = -x \rightarrow \forall y\ (\neg Cy,x \equiv Py,-x)]$

E.g., if a sign y is not connected to a sign x, then y is a part of the complement of x.

T1.37 $\forall x\ (\exists z\ z = -x \rightarrow \neg Cx,-x)$

E.g., a sign x can never be connected to its complement.

T1.38 $\forall x\ [\exists z\ z = -x \rightarrow \forall y\ (x = y \rightarrow -x = -y)]$

E.g., if two signs x and y are connected to one another, then their complements are connected, too.

T1.39 $\forall x\ (\exists z\ z = -x \rightarrow \forall y\ Py,x+-x)$

E.g., if a sign z = -x, then y is a part of the union of x and its complement.

T1.40 $\forall x \forall y\ [(\exists z\ z = -x \wedge \exists z\ z = -y) \rightarrow Px,y \equiv P-y,-x)]$

E.g., if a sign x is a part of a sign y, then the complement –y is a part of the complement –x, too.

T1.41 $\forall x\ (\exists z\ z = -x \rightarrow x + -x = a^*)$

E.g., the union of all signs x and their complements –x is even to the semiotic quasi-Boolean universal.

T1.42 $\forall x \forall y\ (\exists z\ z = x \wedge y \equiv Ox,y)$

E.g., let be x = (3.1 2.2 1.3) and y = (3.1 2.2 12), then x overlaps y, since the intersection x ∧ y = (3.1 2.2).

T1.43 $\forall x \forall y\ (\exists w\ w = x \wedge y \rightarrow \forall z\ \{Cz,x{\wedge}y \equiv \exists w\ [(Pw,x \wedge Pw,y) \wedge Cz,w]\})$

E.g., x = (3.1 2.2 1.3), y = (3.1 2.2 1.2), z = (3.1 2.1 1.1). Then, w = (3.1 2.2), and C((3.1 2.1 1.1), (3.1 2.2)) = (P((3.1 2.2), (3.1 2.2 1.3)) ∧ P((3.1 2.2), (3.1 2.2 1.2))) ∧ C((3.1 2.1 1.1), (3.1 2.2)).

T1.44 $\forall x \forall y\ \{\exists w\ w = x \wedge y \rightarrow \forall z\ [Cz,x{\wedge}y \rightarrow (Cz,x \wedge Cz,y)]\}$

E.g., if a sign z is connected to the intersection of two signs x and y, then z is both connected to x and to y.

T1.45 $\forall x \forall y\ \{\exists w\ w = x \wedge y \rightarrow \forall z\ [(Pz,x \wedge Pz,y) \equiv Pz,x{\wedge}y]\}$

E.g., if a sign z is a part of a sign x and also a part of a sign y, then z is a part of the intersection of x and y, too.

T1.46 $\forall x \forall y\ \{[(\exists z\ z = -x \wedge \exists z\ z = -y) \wedge \exists z\ z = x \wedge y] \rightarrow x \wedge y = -(-x + -y)\}$

E.g., if a sign z can take the values of –x, -y, and x ∧ y, then the intersection of x and y is even to the complement of the union of the complements of x and y.

T1.47 $\forall x\ x \wedge x \equiv x$

E.g., (3.1 2.1 1.3) ∧ (3.1 2.1 1.3) = (3.1 2.1 1.3).

T1.48 $\forall x \forall y\ (\exists z\ z = x \wedge y \rightarrow x \wedge y = y \wedge x$

E.g., (3.1 2.1 1.3) ∧ (3.1 2.2 1.3) = (3.1 2.2 1.3) ∧ (3.1 2.1 1.3).

T1.49 $\forall x \forall y \forall z\ \{[((\exists w\ w = x \wedge y) \wedge (\exists w\ w = y \wedge z) \wedge (\exists w\ w = (x \wedge y) \wedge z] \rightarrow (x \wedge y) \wedge z = x \wedge (y \wedge z)\}$

E.g., ((3.1 2.2 1.3) ∧ (3.1 2.2 1.2)) ∧ (3.1 2.3 1.3) = (3.1 2.2 1.3) ∧ ((3.1 2.2 1.2) ∧ (3.1 2.3 1.3)).

T1.50 $\forall x \forall y\ (\exists z\ z = x \wedge y \rightarrow Px{\wedge}y,x)$

E.g., the intersection of two signs x and z are part of the sign x.

T1.51 $\forall x \forall y\ [\exists z\ z = x \wedge y \rightarrow (Px,y \equiv x = x \wedge y)]$

E.g., if a sign x is a part of a sign y, then the intersection of x and y is a part of y.

T1.52 $\forall x \forall y\ [\exists w\ w = x \wedge y \rightarrow \forall z\ (Px,y \rightarrow Px{\wedge}z,y)]$

E.g., if a sign x is a part of a sign y, then the intersection of x with a sign z is a part of y, too.

T1.53 $\forall x \forall z\ [\exists w\ w = x \wedge z \rightarrow \forall y\ (x = y \rightarrow x \wedge z = y \wedge z)]$

E.g., if sign x is substituted by a sign y, then the intersection of x and z is even to the intersection of y and z.

T1.54 $\forall x \forall y\ \{\exists w\ w = x \wedge y \rightarrow \forall z\ [NTPz,x{\wedge}y \rightarrow (NTPz,x \wedge NTPz,y)]\}$

E.g., a sign z is a non-tangential part of the intersection of two signs x and y, if z is a non-tangential part of both x and y.

T1.55 $\forall x\ x \wedge a^* = x$

E.g., the intersection of a sign-class x with all other nine sign-classes of the set of the ten sign-classes is the set containing the sign-class x.

T1.56 $\forall x \forall y\ \{[(\exists z\ z = \text{-}x) \wedge (\exists z\ z = \text{-}y) \wedge \neg ECx,\text{-}y] \rightarrow (\text{-}x + y = a^* \equiv Px,y)\}$

E.g., if there is no external connection of a sign x to a the complement of a sign y, then the union of the complement of the sign x and the sign y is even to a*, i.e. the semiotic quasi-Boolean universal, which statement means the same like that x is a part of y.

4. Quasi-topological calculus

Following Clarke (1981, p. 212), we shall now introduce the quasi-topological operators, ix for the interior of x, cx for the closure of x, and ex for the exterior of x, and quasi-topological predicates as OPx for "x is open", and CLx for "x is closed":

D2.1 $ix := f'\{y: NTPy,x\}$

D2.2 $cx := f'\ \{y: \neg Cy,i\text{-}x\}$

D2.3 $ex := f'\ \{y: NTPy,\text{-}x\}$

D2.4 $OPx := x = ix$

D2.5 $CLx := x = cx$

Further, we need the following axiom:

A2.1 $\forall x\ (\exists z\ NTPz,x \wedge \forall y\ \forall z\ \{[(Cz,x \rightarrow Oz,x) \wedge (Cz,y \rightarrow Oz,y)] \rightarrow (Cz,x{\wedge}y \rightarrow Oz,x{\wedge}y)\})$

E.g., if the connection of two signs z and x implies the overlap of z over x, and if the connection of two signs z and y implies the overlap of z over y, then the connection of z and the intersection of x and y implies the overlap of z and the intersection of x and y.

In displaying the following 45 theorems, we will again follow Clarke (1981, pp. 213 ss.):

T2.1 $\forall x \, \exists y \; y = ix$

E.g., the distinction between interior, exterior, closure, open and closes sets is valid for semiotic sets, too.

T2.2 $\forall x \forall y \; [Cy,ix \equiv \exists z \; (NTPz,x \wedge Cy,z)]$

E.g., if a sign y is connected to the interior of a sign x, then there is a z such that z is a non-tangential part of x, and y is connected to z.

T2.3 $\forall x \forall y \; (NTPy,x \rightarrow Py,ix)$

E.g., if a sign y is a non-tangential part of a sign x, the y is a part of the interior of x.

T2.4 $\forall x \; Pix,x$

E.g., the interior of a sign x is a part of x.

T2.5 $\forall x \forall y \; (Cy,ix \rightarrow Oy,x)$

E.g., if a sign y is connected to the interior of a sign x, then y overlaps x.

T2.6 $\forall x \forall y \; (ECy,x \rightarrow \neg Cy,ix)$

E.g., if a sign y is externally connected to a sign x, then y is not connected to the interior of x.

T2.7 $\forall x \forall y \; (ECy,x \rightarrow \neg ECy,ix)$

E.g., if a sign y is externally connected to a sign x, then y is not externally connected to the interior of x.

T2.8 $\forall x \forall y \; (Py,ix \rightarrow Py,x)$

E.g., if a sign y is connected to the interior of x, then y is a part of x.

T2.9 $\forall x \; NTPix,x$

E.g., the interior of a sign x is a non-tangential part of x.

T2.10 $\forall x \neg TPix,x$

E.g., the interior of a sign x is not a tangential part of x.

T2.11 $\forall x\forall y\ (Py,ix \equiv NTPy,x)$

E.g., the statement that a sign y is a part of the interior of a sign x is equivalent to the statement that y is a non-tangential part of x.

T2.12 $\forall x\forall y\forall z\ [(NTPx,y \wedge Cz,x) \rightarrow Cz,iy]$

E.g., if a sign x is a non-tangential part of a sign y, and a sign z is connected to x, then z is connected to the interior of y.

T2.13 $\forall x\forall y\forall z\ [(NTPx,y \wedge Oz,x) \rightarrow Oz,iy]$

E.g., if a sign x is a non-tangential part of a sign y, and if a sign z overlaps x, then z overlaps the interior of y.

T2.14 $\forall x\forall y\ (Px,y \rightarrow Pix,iy)$

E.g., if a sign x is a part of a sign y, then the interior of x is (also) a part of the interior of y.

T2.15 $\forall x\forall y\ (x = y \rightarrow ix = iy)$

E.g., if two signs x and y are even, then their interiors are even, too.

T2.16 $\forall x\ ix + x = x$

E.g., the union of a sign and its interior is this sign.

T2.17 $\forall x\ ix \wedge x = ix$

E.g., the intersection of a sign and its interior is this interior.

T2.18 $\forall x\ (NTPx,x \equiv ix = x)$

E.g., the statement that a sign x is a non-tangential part of itself is equivalent to the statement that x is even to its interior.

T2.19 $\forall x\forall y\ (Ox,y \equiv Oix,ix)$

E.g., the statement that a sign x overlaps a sign y is equivalent to the statement that the interior of a sign x overlaps the interior of a sign y.

T2.20 $\forall x \forall y\ (Ox,y \equiv Ox,iy)$

E.g., the statement that a sign x overlaps a sign y is equivalent to the statement that x overlaps the interior of y.

T2.21 $\forall x \forall y\ (Cx,iy \equiv Ox,y)$

E.g., the statement that a sign x is connected to the interior of a sign y is equivalent to the statement that x overlaps y.

T2.22 $\forall x \forall y\ (Cx,iy \equiv Ox,iy)$

E.g., the statement that a sign x is connected to the interior of a sign y is equivalent to the statement that x overlaps the interior of y.

T2.23 $\forall x \forall y\ (Cix,iy \equiv Oix,iy)$

E.g., the statement that the interior of a sign x is connected to the interior of a sign y is equivalent to the statement that the interior of x overlaps the interior of y.

T2.24 $\forall x \forall y\ (\exists z\ z = x \wedge y \equiv \exists z\ z = ix \wedge iy)$

E.g., two sign x and y intersect iff their interiors intersect.

T2.25 $\forall x \forall y\ \neg ECx,iy$

E.g., a sign x cannot be externally connected to the interior of a sign y.

T2.26 $\forall x\ Pix,iix$

E.g., the interior of a sign x is part of itself.

T2.27 $\forall x\ iix = ix$

E.g., the interior of the interior of a sign x is even to the interior of this sign x.

T2.28 $ia^* = a^*$

E.g., the interior of the semiotic universal is even to the universal.

T2.29 $\forall x \forall y\ (\exists z\ z = x \wedge y \rightarrow Pix{\wedge}iy,x{\wedge}y)$

E.g., if two signs x and y intersect, then the intersection of their interiors is a part of the intersection of the signs.

T2.30 $\forall x \forall y\ (\exists z\ z = x \wedge y \rightarrow Pi(x \wedge y),ix \wedge iy)$

E.g., if two signs x and y intersect, then the interior of their intersection is a part of the intersection of their interiors.

T2.31 $\forall x \forall y\ \{\exists w\ w = x \wedge y \rightarrow \forall z\ [NTPz,x \wedge NTPz,y) \equiv NTPz,x \wedge y]\}$

E.g., if a sign z is a non-tangential part of a sign x and a non-tangential part of a sign y, then it is also a non-tangential part of the intersection of x and y.

T2.32 $\forall x \forall y\ (\exists z\ z = x \wedge y \rightarrow ix \wedge iy = i(x \wedge y))$

E.g., the intersection of the interior of a sign x and the interior of a sign y is the same as the interior of the intersection of these two signs.

T2.33 $\forall x\ (\exists z\ z = cx \equiv \exists y\ \neg Cy,i\text{-}x)$

E.g., if a sign z is the closure of a sign x, this means that there is a sign y such that y is not connected to the interior of the complement of x.

T2.34 $\forall x\ \{\exists z\ z = cx \rightarrow \forall w\ w\ [Cw,cx \equiv \exists y\ \neg Cy,i\text{-}x \wedge Cw,y)]\}$

E.g., if a sign w is connected to the closure of a sign x, then this means that a sign y is not connected to the interior of the complement of x, and w is connected to y.

T2.35 $\forall x\ (\exists z\ z = \text{-}x \rightarrow \exists z\ z = cx)$

E.g., a sign that has a complement, has also a closure.

T2.36 $\forall x\ [\exists z\ z = \text{-}x \rightarrow \forall w\ (Cw,cx \equiv \neg NTPw,\text{-}x)]$

E.g., $\forall x\ [\exists z\ z = \text{-}x \rightarrow \forall w\ (Cw,cx \equiv \neg NTPw,\text{-}x)]$

E.g., if a sign w is connected to the closure of a sign x, then w cannot be a non-tangential part of the complement of x.

T2.37 $\forall x\ (\exists z\ z = \text{-}x \rightarrow cx = \text{-}i\text{-}x)$

E.g., if a sign z is the complement of a sign x, then the closure of x is even to the complement of the interior of the complement of x.

T2.38 $\forall x\ (\exists z\ z = \text{-}x \rightarrow i\text{-}x = \text{-}cx)$

E.g., if a sign z ist the complement of a sign x, then the interior of the complement of x is even to the complement of the closure of x.

T2.39 $\forall x\ (\exists z\ z = -x \rightarrow c\text{-}x = -ix)$

E.g., if a sign z is the complement of a sign x, then the closure of the complement of x is even to the complement of the interior of x.

T2.40 $\forall x\ (\exists z\ z = -x \rightarrow ix = -c\text{-}x)$

E.g., if a sign z is the complement of a sign x, then the interior of x is even to the complement of the closure of the complement of x.

T2.41 $\forall x\ (\exists z\ z = -x \rightarrow Px,cx)$

E.g., if a sign z is the complement of a sign x, then x is a part of its closure.

T2.42 $\forall x\ (\exists z\ z = -x \rightarrow ccx = cx)$

E.g., if a sign z is the complement of a sign x, then the complement of the complement of x is even to the (simple) complement of x.

T2.43 $\forall x \forall y\ \{[(\exists z\ z = -x \wedge \exists z\ z = -y) \wedge (\exists z\ z = -x \wedge -y)] \rightarrow cx + cy = c(x+y)\}$

E.g., the union of the complement of a sign a with the complement of a sign y is even to the complement of the union of x and y.

T2.44 $\forall x \forall y\ [(\exists z\ z = -x \wedge \exists z\ z = -y) \rightarrow Px,y \rightarrow Pcx,cy)]$

E.g., if a sign x is a part of a sign y, then the complement of x is a part of the complement of y, too.

T2.45 $\forall x\ (\exists z\ z = -x \rightarrow ex = i\text{-}x)$

E.g., if a sign z is the complement of a sign x, then the exterior of x is even to the interior of the complement of x.

The calculi of extensional logical sign connections established in this study for a semiotic mereology, a semiotic quasi-Boolean algebra and a semiotic quasi-topology complete the system of the purely semiotic sign connections established in Toth (2008a).

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