Strict weak orderings in semiotics

1. A strict weak ordering is a binary relation $<$ on a set $S$ that is a strict partial order, i.e. a transitive relation that is irreflexive, or equivalently, that is asymmetric, in which the relation “neither $a < b$ nor $b < a$” is transitive. The equivalence classes of this “incomparability relation” partition the elements of $S$, and are totally ordered by $<$. Conversely, any total order on a partition of $S$ gives rise to a strict weak ordering in which $x < y$ if and only if there exists sets $A$ and $B$ in the partition with $x$ in $A$, $y$ in $B$, and $A < B$ in the total order (Roberts 1979).

A strict weak ordering has the following properties. For all $x$ and $y$ in $S$,

- For all $x$, it is not the case that $x < x$ (irreflexivity).
- For all $x \neq y$, if $x < y$ then it is not the case that $y < x$ (asymmetry).
- For all $x$, $y$, and $z$, if $x < y$ and $y < z$ then $x < z$ (transitivity).
- For all $x$, $y$, and $z$, if $x$ is incomparable with $y$, and $y$ is incomparable with $z$, then $x$ is incomparable with $z$ (transitivity of equivalence) $\equiv$ If $x < y$, then for all $z$ either $x < z$ or $z < y$ or both

2. As a first example, we show the 13 possible strict weak orders on the set $SR_{3,3} = \{.1, .2, .3\}$, or simplified $\{1, 2, 3\}$, of the triadic-trichotomic sign relation:

\[
\{\{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{3\}, \{2\}\}, \{\{2\}, \{1\}, \{3\}\}, \{\{3\}, \{1\}, \{2\}\}, \{\{2\}, \{3\}, \{1\}\}, \{\{3\}, \{2\}, \{1\}\}, \{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}, \{\{3\}, \{1, 2\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{2, 3\}, \{1\}\}, \{\{1, 2, 3\}\}\}
\]

that can be displayed with the following graph:
3. Strict weak orders are very closely related to total preorders or (non-strict) weak orders, and the same mathematical concepts can be modeled equally well with total preorders. A total preorder or weak order is a preorder that is total; that is, no pair of items is incomparable. A total preorder $\leq$ satisfies the following properties:

- For all $x$, $y$, and $z$, if $x \leq y$, and $y \leq z$, then $x \leq z$ (transitivity).
- For all $x$ and $y$, $x \leq y$ or $y \leq x$ (totality).
- Hence: For all $x$, $x \leq x$ (reflexivity).

A total order is a total preorder which is antisymmetric, in other words, which is also a partial order (Roberts 1979). The number of total preorders is given by the Fubini numbers or ordered Bell numbers:

<table>
<thead>
<tr>
<th>n</th>
<th>all</th>
<th>trans.</th>
<th>refl.</th>
<th>preor.</th>
<th>part. order</th>
<th>total preorder</th>
<th>total order</th>
<th>equiv. rel.</th>
</tr>
</thead>
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<td>1--</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2--</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
<td>13</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
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<td>512</td>
<td>171</td>
<td>64</td>
<td>29</td>
<td>19</td>
<td>13</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
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<td>65'536</td>
<td>3'994</td>
<td>4'096</td>
<td>355</td>
<td>219</td>
<td>75</td>
<td>24</td>
<td>15</td>
</tr>
</tbody>
</table>
We have already shown the 13 total preorders for the set \{1., .2., .3.\} with \(n = 3\). For \(n = 4\), for which we can take as examples \(\text{SR}_{4,3} = \{0., .1., .2., .3.\}\) or \(\text{SR}_{4,4} = \{0., .1., .2., .3.\}\) (cf. Toth 2008a), we have 75 total preorders:

- 1 partition of 4 sets, giving 1 total preorder, i.e. each element is related to each element:
  \{0, 1, 2, 3\}
- 7 partitions of 2 sets, giving 14 total preorders:
  \{\{0, 3\}, \{1, 2\}\}, \{\{1, 2\}, \{0, 3\}\}, \{\{0\}, \{1, 2, 3\}\}, \{\{1, 2, 3\}, \{0\}\}, \{\{0, 1, 3\}, \{2\}\},
  \{\{2\}, \{0, 1, 3\}\}, \{\{0, 2\}, \{1, 3\}\}, \{\{1, 3\}, \{0, 2\}\}, \{\{0, 1, 2\}, \{3\}\}, \{\{3\}, \{0, 1, 2\}\},
  \{\{0, 2, 3\}, \{1\}\}, \{\{1\}, \{0, 2, 3\}\}, \{\{0, 1\}, \{2, 3\}\}, \{\{2, 3\}, \{0, 1\}\}
- 6 partitions of 3 sets, giving 36 total preorders:
  \{\{0\}, \{1, 2\}, \{3\}\}, \{\{0\}, \{3\}, \{1, 2\}\}, \{\{1, 2\}, \{0\}, \{3\}\}, \{\{1, 2\}, \{3\}, \{0\}\},
  \{\{3\}, \{1, 2\}, \{0\}\}, \{\{3\}, \{0\}, \{1, 2\}\}
  \{\{0, 3\}, \{1\}, \{2\}\}, ...
  \{\{0\}, \{1, 3\}, \{2\}\}, ...
  \{\{0, 2\}, \{1\}, \{3\}\}, ...
  \{\{0, 1\}, \{2\}, \{3\}\}, ...
  \{\{0\}, \{1\}, \{2, 3\}\}, ...
- 1 partition of 1 set, giving 24 total preorders, i.e. the total orders:
  \{\{1\}\}, \{\{2\}\}, \{\{3\}\}, \{\{4\}\}\} and all permutations

The number of ordered partitions \(T_n\) of \{1, 2, ..., \(n\)\} is calculated recursively by

\[
T_n = \sum_{i=0}^{n-1} T_i \cdot T_{n-i}
\]

A strict weak order that is trichotomous is called a strict total order, i.e. exactly one of the relations \(a < b\), \(b < a\), \(a = b\) holds. E.g., for the set of the triadic-trichotomic sign classes based on \(\text{SR}_{3,3} = (3.a \ 2.b \ 1.c)\) with \(a \leq b \leq c\), we get the following sets of pairs of dyads:

\((a < b)\):
\{(3.1, 2.2), (3.1, 2.3), (3.2, 2.3), (2.1, 1.2), (2.1, 1.3), (2.2, 1.3)\}

\((a = b)\):
\{(3.1, 2.1), (3.2, 2.2), (3.3, 2.3), (2.1, 1.1), (2.2, 1.2), (2.3, 1.3)\}

However, the relation \((b < a)\) does not hold in \(\text{SR}_{3,3}\) as long as the trichotomic semiotic inclusion order is valid; therefore, we find this type of order only in the 17 complementary sign classes out of the total amount of 27 triadic-trichotomic sign classes (cf. Toth 2008b)

\((b < a)\):
\{(3.2 \ 2.1), (3.3, 2.1), (3.3, 2.2), (2.2, 1.1), (2.3, 1.1), (2.3, 1.2)\}.

Moreover, this order type is present as main diagonal in the semiotic matrix over \(\text{SR}_{3,3}\):
This so-called Genuine Category Class (cf. Bense 1992, pp. 27 ss.) (3.3 2.2 1.1) has trichotomic order (3.a 2.b 1.c) with $a > b > c$ which is at the same time trichotomous. In the set of the 10 sign classes, it shares trichotousness only with the sub-set of the homogeneous sign classes on the one side \{(3.1 2.1 1.1), (3.2 2.2 1.2), (3.3 2.3 1.3) with trichotomic order $a = b = c)\}$ and with the eigen-real sign class (3.1 2.2 1.3) with trichotomic order $(a < b < c)$ on the other side; the other 6 sign classes are of mixed trichotomic order and thus not trichotomous.

Bibliography

Roberts, Fred S., Measurement theory. Addison-Wesley 1979
Toth, Alfred, Tetradic, triadic, and dyadic sign classes. Ch. 44 (2008a)
Toth, Alfred, Homeostasis in semiotic systems. Ch. 4 (vol. I) (2008b)

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