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Triangulations of the semiotic hexagon and semiotic Catalan numbers

1. The Catalan numbers (1, 2, 5, 14, 42, ...) arise in at least 66 combinatorial problems (cf. Stanley 1999, pp. 219-229). They are computed by the formula

$$\frac{\binom{2n}{n}}{n+1}$$

In the present study, we shall show that Catalan numbers indicate the number of ways a semiotic hexagon can be cut into 4 triangles. Further, we show that Catalan numbers predict the number of paths of length 2n through an n × n grid that do not rise above the main diagonal (cf. also Toth 2008a, b, c, d).

2. Each sign class of the form

(3.a 2.b 1.c)

has the following 6 permutations:

(3.a 2.b 1.c) (2.b 3.a 1.c) (1.c 3.a 2.b)
(3.a 1.c 2.b) (2.b 1.c 3.a) (1.c 2.b 3.a)

Correspondingly, each dual reality thematic of the form

(c.1 b.2 a.3)

has the following 6 permutations (cf. Toth 2008, pp. 177 ss.):

(c.1 b.2 a.3) (b.2 c.1 a.3) (a.3 c.1 b.2)
(c.1 a.3 b.2) (b.2 a.3 c.1) (a.3 b.2 c.1)

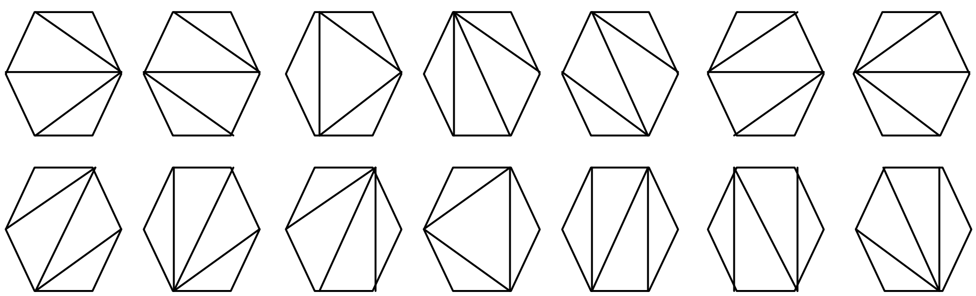
Therefore, the 6 partitions of a sign class or reality thematic can be ordered in a semiotic hexagon, e.g., in the following clockwise order:

(1.c 2.b 3.a) (3.a 2.b 1.c)

(1.c 3.a 2.b) (3.a 1.c 2.b)

(2.b 1.c 3.a) (2.b 3.a 1.c)

Now, the fourth Catalan number tells us that a hexagon can be triangulated in 14 different ways:



By these triangulations, we get the following 14 · 3 sign connections (from the left of the top to the right of the bottom):

1. (1.c 2.b 3.a) (3.a 1.c 2.b) (3.a 1.c 2.b)
   (3.a 1.c 2.b) (1.c 3.a 2.b) (2.b 1.c 3.a) } ≡

[[α, (c.b)], [β, (b.a)]], [[α°β°, (a.c)], [α, (c.b)]], [[α°β°, (a.c)], [α, (c.b)]]
[[α°β°, (a.c)], [α, (c.b)]], [[βα, (c.a)], [β°, (a.b)]], [[α°, (b.c)], [βα, (c.a)]]

2. (1.c 2.b 3.a) (3.a 1.c 2.b) (2.b 3.a 1.c)
   (3.a 1.c 2.b) (1.c 3.a 2.b) (1.c 3.a 2.b) } ≡

[[α, (c.b)], [β, (b.a)]], [[α°β°, (a.c)], [α, (c.b)]], [[β, (b.a)], [α°β°, (a.c)]]
[[α°β°, (a.c)], [α, (c.b)]], [[βα, (c.a)], [β°, (a.b)]], [[βα, (c.a)], [β°, (a.b)]]

3. (1.c 2b. 3.a) (3.a 1.c 2.b) (2.b 1.c 3.a)
   (3.a 1.c 2.b) (2.b 1.c 3.a) (1.c 2.b 3.a) } ≡

[[α, (c.b)], [β, (b.a)]], [[α°β°, (a.c)], [α, (c.b)]], [[α°, (b.c)], [βα, (c.a)]]
[[α°β°, (a.c)], [α, c.b]], [[α°, (b.c)], [βα, (c.a)]], [[α, (c.b)], [β, (b.a]]

4. (1.c 2.b 3.a) (1.c 2.b 3.a) (1.c 2.b 3.a)
(3.a 1.c 2.b) (2.b 3.a 1.c) (2.b 1.c 3.a) ≡

[[α, (c.b)], [β, (b.a)]], [[α, (c.b)], [β, (b.a)]], [[α, (c.b)], [β, (b.a)]]
[[α°β°, (a.c)], [α, (c.b)]] [[β, (b.a)], [α°β°, (a.c)]], [[α°, (b.c)], [βα, (c.a)]]

5. (1.c 2.b 3.a) (1.c 2.b 3.a) (2.b 3.a 1.c)
(3.a 1.c 2.b) (2.b 3.a 1.c) (1.c 3.a 2.b) ≡

[[α, (c.b)], [β, (b.a)]], [[α, (c.b)], [β, (b.a)]], [[β, (b.a)], [α°β°, (a.c)]]
[[α°β°, (a.c)], [α, (c.b)]], [[β, (b.a)], [α°β°, (a.c)]], [[βα, (c.a), [β°, (a.b)]]

6. (3.a 2.b 1.c) (3.a 1.c 2.b) (3.a 1.c 2.b
(1.c 3.a 2.b) (1.c 3.a 2.b) (2.b 1.c 3.a) ≡

[[β°, (a.b)], [α°, (b.c)]], [[α°β°, (a.c)], [α, (c.b)]], [[α°β°, (a.c)], [α, (c.b)]]
[[βα, (c.a)], [β°, (a.b)]], [βα, (c.a)], [β°, (a.b)]], [[α°, (b.c)], [βα, (c.a)]]

7. (3.a 2.b 1.c) (3.a 1.c 2.b) (2.b 3.a 1.c)
(1.c 3.a 2.b) (1.c 3.a 2.b) (1.c 3.a 2.b) ≡

[[β°, (a.b)], [α°, (b.c)]], [[α°β°, (a.c)], [α, (c.b)]], [[β, (b.a)], [α°β°, (a.c)]]
[[βα, (c.a)], [β°, (a.b)]], [[βα, (c.a)], [β°, (a.b)]], [[βα, (c.a)], [β°, (a.b)]]

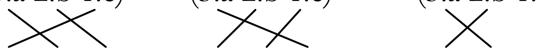
8. (3.a 2.b 1.c) (3.a 2.b 1.c) (3.a 1.c 2.b)
(1.c 3.a 2.b) (2.b 1.c 3.a) (2.b 1.c 3.a) ≡

[[β°, (a.b)], [α°, (b.c)]], [[β°, (a.b)], [α°, (b.c)]], [[α°β°, (a.c)], [α, (c.b)]]
[[βα, (c.a)], [β°, (a.b)]], [[α°, (b.c)], [βα, (c.a)]], [[α°, (b.c)], [βα, (c.a)]]

9. (1.c 2.b 3.a) (3.a 2.b 1.c) (3.a 1.c 2.b)
(2.b 1.c 3.a) (2.b 1.c 3.a) (2.b 1.c 3.a) ≡

[[α, (c.b)], [β, (b.a)]], [[β°, (a.b)], [α°, (b.c)]], [[α°β°, (a.c)], [α, (c.b)]]
[[α°, (b.c)], [βα, (c.a)]], [[α°, (b.c)], [βα, (c.a)]], [α°, (b.c)], [βα, (c.a)]]

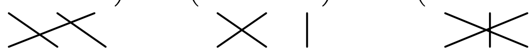
10. (3.a 2.b 1.c) (3.a 2.b 1.c) (3.a 2.b 1.c)



(1.c 3.a 2.b) (2.b 1.c 3.a) (2.b 3.a 1.c) ≡

[[β°, (a.b)], [α°, (b.c)]], [[β°, (a.b)], [α°, (b.c)]], [[β°, (a.b)], [α°, (b.c)]]
[[βα, (c.a)], [β°, (a.b)]], [[α°, (b.c)], [βα, (c.a)]], [[β, (b.a), [α°β°, (a.c)]]

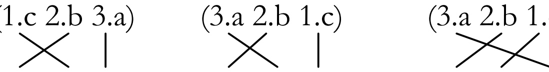
11. (3.a 2.b 1.c) (3.a 2.b 1.c) (2.b 3.a 1.c)



(1.c 3.a 2.b) (2.b 3.a 1.c) (1.c 3.a 2.b) ≡

[[β°, (a.b)], [α°, (b.c)]], [[β°, (a.b)], [α°, (b.c)]], [[β, (b.a)], [α°β°, (a.c)]]
[[βα, (c.a)], [β°, (a.b)]], [[β, (b.a)], [α°β°, (a.c)]], [[βα, (c.a)], [β°, (a.b)]]

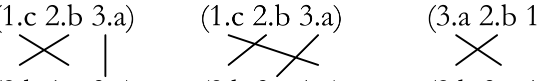
12. (1.c 2.b 3.a) (3.a 2.b 1.c) (3.a 2.b 1.c)



(2.b 1.c 3.a) (2.b 3.a 1.c) (2.b 1.c 3.a) ≡

[[α, (c.b)], [β, (b.a)]], [[β°, (a.b)], [α°, (b.c)]], [[β°, (a.b)], [α°, (b.c)]]
[[α°, (b.c)], [βα, (c.a)]], [[β, (b.a)], [α°β°, (a.c)]], [[α°, (b.c)], [βα, (c.a)]]

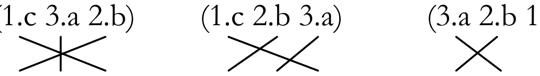
13. (1.c 2.b 3.a) (1.c 2.b 3.a) (3.a 2.b 1.c)



(2.b 1.c 3.a) (2.b 3.a 1.c) (2.b 3.a 1.c) ≡

[[α, (c.b)], [β, (b.a)]], [[α, (c.b)], [β, (b.a)]], [[β°, (a.b)], [α°, (b.c)]]
[[α°, (b.c)], [βα, (c.a)]], [[β, (b.a)], [α°β°, (a.c)]], [[β, (b.a)], [α°β°, (a.c)]]

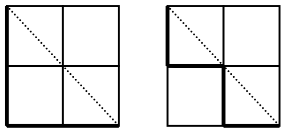
14. (1.c 3.a 2.b) (1.c 2.b 3.a) (3.a 2.b 1.c)



(2.b 3.a 1.c) (2.b 3.a 1.c) (2.b 3.a 1.c) ≡

[[βα, (c.a)], [β°, (a.b)]], [[α, (c.b)], [β, (b.a)]], [[β°, (a.b)], [α°, (b.c)]]
[[β, (b.a)], [α°β°, (a.c)]], [[β, (b.a)], [α°β°, (a.c)]], [[β, (b.a)], [α°β°, (a.c)]]

3. Since Catalan numbers also indicate the number of paths of length 2n through an $n \times n$ grid underneath of the main diagonal, we have 2 paths of length 4 in a 2×2 grid:



If these 2×2 grids are representing $SR_{3,3}$ or $SR_{4,3}$, we get:

1. ((1.1, 2.1), (2.1, 3.1), (3.1, 3.2), (3.2, 3.3)) ≡ [[α, id1], [β, id1], [id3, α], [id3, β]]
2. ((1.1, 2.1), (2.1, 2.2), (2.2, 3.2), (3.2, 3.3)) ≡ [[α, id1], [id2, α], [β, id2], [id3, β]]

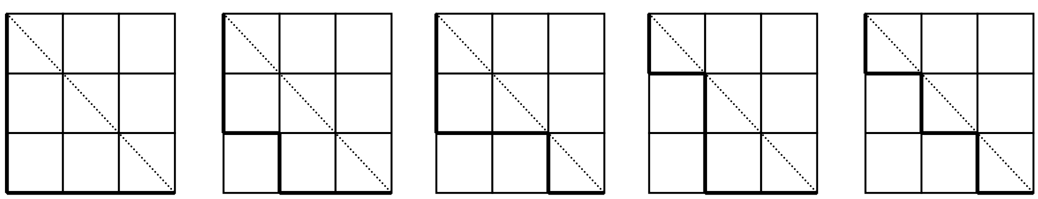
If they represent the upper part of the matrix of $SR_{4,4}$, we have:

3. ((0.0, 1.0), (1.0, 2.0), (2.0, 2.1), (2.1, 2.2)) ≡ [[γ, id0], [α, id0, [id2, γ], [id2, α]]
4. ((0.0, 1.0), (1.0, 1.1), (1.1, 2.1), (2.1, 2.2)) ≡ [[γ, id0], [id1, γ], [α, id1], [id2, α]]

and if they represent the lower part of the matrix of $SR_{4,4}$, then:

5. ((1.0, 2.0), (2.0, 3.0), (3.0, 3.1), (3.1, 3.2)) ≡ [[α, id0], [β, id0], [id3, γ], [id3, α]]
6. ((1.0, 2.0), (2.0, 2.1), (2.1, 3.1), (3.1, 3.2)) ≡ [[α, id0], [id2, γ], [β, id1], [id3, α]]

In a 3×3 grid, we have 5 paths of length 6:



Since a 3×3 grid represents the matrix of $SR_{4,4}$, we have:

7. ((0.0, 1.0), (1.0, 2.0), (2.0, 3.0), (3.0, 3.1), (3.1, 3.2), (3.2, 3.3)) ≡
   [[γ, id0], [α, id0], [β, id0], [id3, γ], [id3, α], [id3, β]]

8. ((0.0, 1.0), (1.0, 2.0), (2.0, 2.1), (2.1, 3.1), (3.1, 3.2), (3.2, 3.3)) ≡
   [[γ, id0], [α, id0], [id2, γ], [β, id1], [id3, α], [id3, β]]

9. ((0.0, 1.0), (1.0, 2.0), (2.0, 2.1), (2.1, 2.2), (2.2, 3.2), (3.2, 3.3)) ≡
   [[γ, id0], [α, id0], [id2, γ], [id2, α], [β, id2], [id3, β]]

10. ((0.0, 1.0), (1.0, 1.1), (1.1, 2.1), (2.1, 3.1), (3.1, 3.2), (3.2, 3.3)) ≡
    [[γ, id0], [id1, γ], [α, id1], [β, id1], [id3, α], [id3, β]]

11. ((0.0, 1.0), (1.0, 1.1), (1.1, 2.1), (2.1, 2.2), (2.2, 3.2), (3.2, 3.3)) ≡
[[γ, id0], [id1, γ], [α, id1], [id2, α], [β, id2], [id3, β]]

For all these semiotic monotonic paths, the diagonal not to be surpassed is (1.1, 2.2, 3.3) and thus the Genuine Category Class (cf. Bense 1992, pp. 27 ss.).

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